COMPUTING THE h-FUNCTION OF THE COMPLEX PLANE WITH A DELETED LINE SEGMENT, WITH AND WITHOUT USING THE PRIME FUNCTION

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The likelihood of a particle launched from a fixed point z_0 in a region Ω initially leaving the area within the distance r of z_0 is the harmonic-measure distribution function, or the h-function. This function is non-decreasing, right-continuous, and takes values on the unit interval [0,1]. The objective of this paper is to validate the h-function formula obtained via three different approaches for a simply connected region Ω formed by deleting a line segment $[-i, i]$ from the complex plane with basepoint z_0=1. To evaluate the h-function, we employ various forms of conformal mappings, harmonic functions, and the prime function. These three approaches vary in their utilization of the prime function, both in terms of presence and methodology. In the first approach, we completely avoid employing the prime function. Rather, the conformal mapping from the unit disc is expressed through the combination of a Joukowski map and a Mobius transformation, with the h-function being determined by extracting the correct angle of view in the unit disc. In the second and third approaches, the prime function is utilized. In the second method, we do not employ the prime function in the conformal mapping from the disc to the region Ω, as it is essentially a Joukowski transformation. In the meantime, the main function is utilized in the Cayley-type map $R(\zeta)$ from the interior of the unit disc D_ζ to the lower half-plane, which is employed in the creation of the harmonic function $\text{Im}[W(\zeta)]$. In the third approach, the prime function is applied twice: initially during the parallel-slit mapping from the unit disc D_ζ to the target region Ω, and subsequently in $R(\zeta)$ and consequently in $\text{Im}[W(\zeta)]$. All three approaches yield identical h-function graphs. These methods provide a check for our h-function formula.

Keywords: h-functions, harmonic measure, conformal maps, prime function

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INTRODUCTION

The study of the h-functions was initiated by Walden and Ward [WW96]. The h-function is always defined for a region Ω together with a fixed point z_0 in Ω, referred to as the basepoint. For a region Ω and a basepoint $z_0 \in \Omega$, the h-function $h(r)$ is given by the probability that a Brownian particle released from z_0, first hits the boundary $\partial\Omega$ of Ω within distance r of z_0. Alternatively, we can also express the h-function via the harmonic measure. The h-function $h(r)$ is given by the harmonic measure of the set $E_r := \partial\Omega \cap \underline{B(z_0, r)}$ in Ω at z_0, where $B(z_0, r)$ is the open ball of radius r centred at z_0. That is, $h(r) = \omega(z_0, E_r, \Omega)$. We call $|z - z_0| = r$, the capture circle of radius r centred at z_0 (hereafter we call the capture circle). Moreover, the h-function can be expressed in terms of the harmonic-measure, the solution of the Dirichlet problem. In other words, $h(r) = u(z_0)$, where $u(z)$ is the unique function such that $\Delta u = 0$ on Ω with $u = 1$ in E_r and $u = 0$ in $\partial\Omega \backslash E_r$.

The h-function always lies between the values 0 and 1, since it represents the probability. Also, it is an increasing, right-continuous function. Moreover, the h-function is zero until the capture circle covers any part of the boundary $\partial\Omega$. Similarly, the h-function takes the value 1 once the capture circle covers the whole part of the boundary $\partial\Omega$.

The h-functions have been extensively studied for numerous simply connected regions from the first paper [WW96] to the present. See the survey article [SW16] for a summary of this progress. Recently, Mahenthiram [Mah22] studied the h-functions and their specific behaviours in variety of newly considered simply connected and multiply connected regions with diverse geometries. ``See also [Mah23] and [Mah]."

The central question we address here is: How to validate the h-functions via different methods. We provide an answer to this question by computing h-function of a simply connected region for which it is possible to compute the h-function $h(r)$ analytically using three different methods. We consider the domain $\Omega = C\backslash[-i, i]$ with basepoint $z_0 = 1$. In these three methods, we obtain the same h-function graph for this domain Ω when the basepoint is $z_0 = 1$. In the first method, we compute the h-function of the domain Ω with z_0 =1 by using the Joukowski map and a Möbius map, but we do not use the prime function in this method. In the second method, we compute the h-function of the domain Ω with $z_0 = 1$ by using the Joukowski map and the prime function. In the second method, we calculate the h-function of the domain Ω with $z_0 = 1$ via the parallel-slit mapping, expressed in terms of the prime function.

METHODOLOGY

In this section, we provide the definitions and useful formulas which are used in this abstract. Here, we use the parallel-slit mapping expressed in terms of the prime function $\omega(\zeta, \alpha)$ to compute the h-function of the simply connected region formed by the complex plane with a deleted slit. Also, we compute the h-function of the same region using a different approach without involving the prime function.

Prime function $\omega(\zeta, \alpha)$ is a key tool to solve problems related with multiply connected regions. For simply connected case, the prime function $\omega(\zeta, \alpha)$ is a simple monomial. That is, $\omega(\zeta, \alpha) = \zeta - \alpha$.

According to [CM06], the following expression was used to derive the general formula for the parallel-slit mapping in terms of the prime functions:

$$f_\theta(\zeta, \alpha) = \left[\frac{\partial}{\partial \alpha} - e^{2i\theta} \frac{\partial}{\partial \bar{\alpha}}\right] \tilde{G}_0(\zeta, \alpha),$$

where $\tilde{G}_0(\zeta, \alpha) = -\log \left\{\frac{1}{|\alpha|} \frac{\omega(\zeta,\alpha)}{\omega(\zeta, 1/\bar{\alpha})}\right\}$ and θ be the angle subtended by the parallel slits with the positive real axis.

In simply connected cases, $\tilde{G}_0(\zeta, \alpha) = -\log \left\{\frac{1}{|\alpha|} \frac{(\zeta-\alpha)}{(\zeta-1/\bar{\alpha})}\right\}$.

Now, $\tilde{G}_0(\zeta, \alpha) = -\log(\zeta - \alpha) + \log|\alpha| + \log(\zeta - 1/\bar{\alpha})$. Since $\alpha\bar{\alpha} = |\alpha|^2$, we have $\tilde{G}_0(\zeta, \alpha) = -\log(\zeta - \alpha) + \frac{1}{2}[\log \alpha + \log \bar{\alpha}] + \log(\zeta - 1/\bar{\alpha})$. Note that $\bar{\alpha}$ is independent of α. Now, by differentiating $\tilde{G}_0(\zeta, \alpha)$ with respect to α, we obtain $\frac{\partial \tilde{G}_0}{\partial \alpha} = \frac{1}{2\alpha} + \frac{1}{\zeta-\alpha}$. Similarly, by differentiating $\tilde{G}_0(\zeta, \alpha)$ with respect to $\bar{\alpha}$, we obtain $\frac{\partial \tilde{G}_0}{\partial \bar{\alpha}} = \frac{1}{2\bar{\alpha}} + \frac{1}{\bar{\alpha}(\zeta\bar{\alpha}-1)}$.

Hence, we obtain

$$f_\theta(\zeta, \alpha) = \frac{1}{2\alpha} + \frac{1}{\zeta-\alpha} - e^{2i\theta}\left(\frac{1}{2\bar{\alpha}} + \frac{1}{\bar{\alpha}(\zeta\bar{\alpha}-1)}\right). \quad \text{(A)}$$

Now, we move our focus to the computation of $h(r)$. Since we have the vertical slit, the angle $\theta = \pi/2$ in (A). Thus, the equation (A) becomes

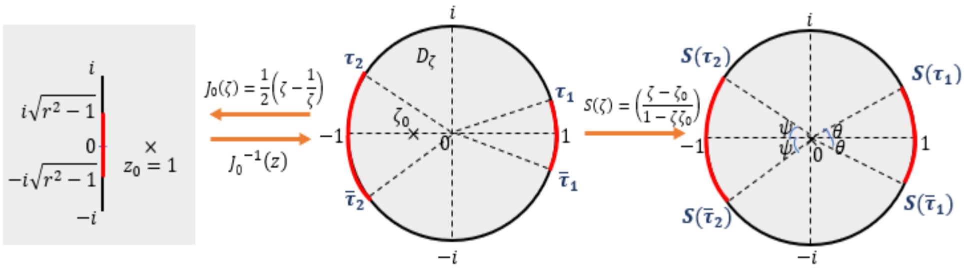
$$f_{\pi/2}(\zeta, \alpha) = \frac{1}{2\alpha} + \frac{1}{\zeta-\alpha} + \left(\frac{1}{2\bar{\alpha}} + \frac{1}{\bar{\alpha}(\zeta\bar{\alpha}-1)}\right) = f(\zeta). \quad \text{(B)}$$

RESULTS AND DISCUSSION

Here, we compute the h-function of the domain Ω with basepoint $z_0 = 2i$ using two different methods.

Method 1: In this method, we compute the h-function of $\Omega = C\backslash[-i, i]$ with basepoint $z_0 = 1$ by using the Joukowski map and a Möbius map, but without using the prime function. Here, we transform the given domain Ω onto the interior of the unit disc with basepoint at the origin. For this, first, we use the Joukowski map $J_0(\zeta) = \frac{1}{2}\left(\zeta - \frac{1}{\zeta}\right)$ to transform the interior of the unit disc onto the given domain Ω. Since the basepoint $z_0 = 1$, the preimage of the basepoint z_0 is $\zeta_0 = 1 - \sqrt{2}$. Now, the inverse Joukowski map J_0^{-1} transforms the domain Ω onto the interior of the unit disc, but the basepoint z_0 is mapped to a nonzero point in the real axis. The Möbius transformation with formula $S(\zeta) = \left(\frac{\zeta-\zeta_0}{1-\zeta\zeta_0}\right)$ maps this new basepoint to the origin without changing the domain. See Figure 1.

Figure 1: Composition of conformal maps from interior of unit disc to domain Ω with basepoint $z_0 = 1$, for regime $r \in [1, \sqrt{2})$. The set E_r and its preimages are shown in red.



In Figure 1, we have $\tau_1 = \sqrt{2-r^2} + i\sqrt{r^2-1}$ and $\tau_2 = -\sqrt{2-r^2} + i\sqrt{r^2-1}$. Moreover, we can show that $S(\tau_1) = \left[\frac{\sqrt{2(2-r^2)}+1}{\sqrt{2}+\sqrt{2-r^2}}\right] + i\left[\frac{\sqrt{r^2-1}}{\sqrt{2}+\sqrt{2-r^2}}\right]$ and $S(\tau_2) = -\left[\frac{\sqrt{2(2-r^2)}-1}{\sqrt{2}-\sqrt{2-r^2}}\right] + i\left[\frac{\sqrt{r^2-1}}{\sqrt{2}-\sqrt{2-r^2}}\right]$. Also, from this figure, the angle of sight from the image of E_r in the unit disc is $2\theta + 2\psi$, where the angles θ and ψ can be found from the expressions $tan\,tan\,\theta = \frac{\sqrt{r^2-1}}{\sqrt{2(2-r^2)}+1}$ and $tan\,tan\,\psi = \frac{\sqrt{r^2-1}}{\sqrt{2(2-r^2)}-1}$. By using the tangent addition formula, we obtain that $tan\,tan\,(\theta+\psi) = \frac{2\sqrt{2(r^2-1)(2-r^2)}}{4-3r^2}$. Hence, the h-function is given by $h(r) = \frac{1}{2\pi}(2\theta + 2\psi) = \frac{1}{\pi} arctan\left\{\frac{2\sqrt{2(r^2-1)(2-r^2)}}{4-3r^2}\right\}$. This formula produces the following h-function graph shown in Figure 2.

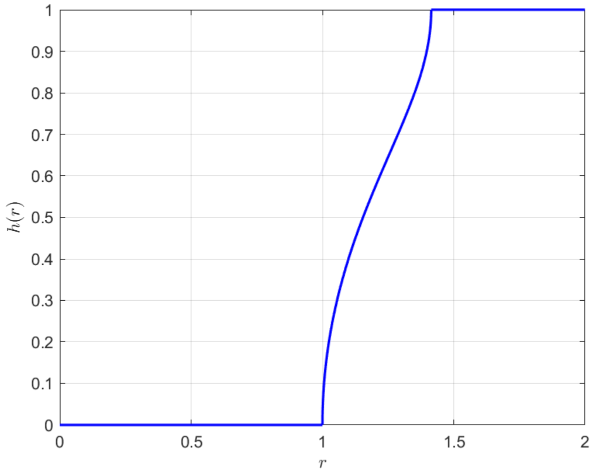


Figure 2: Plot of h-function for domain $\Omega = C\backslash[-i, i]$ with basepoint $z_0 = 1$.

<u>Method 2</u>: In this method, we explain the h-function computation by using the prime function. However, the conformal map does not use the prime function. In this method, we only use the Joukowski map J_0 which is used in Method 1. See Figure 3. Since $z_0 = 1$, its preimage is $\zeta_0 = 1 - \sqrt{2} \in D_\zeta$, where D_ζ is the interior of the unit disc.

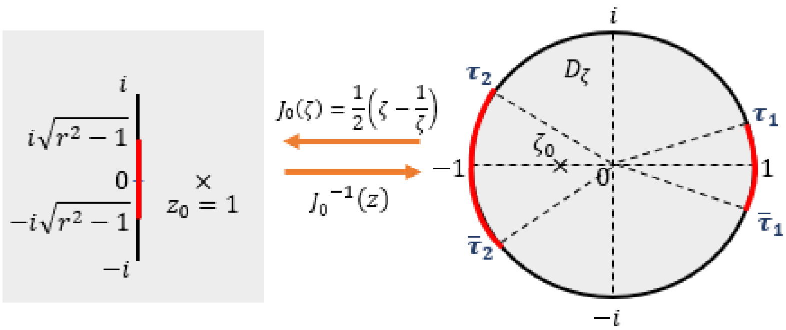


Figure 3: Joukowski map from interior of unit disc to domain Ω with basepoint $z_0 = 1$, for regime $r \in [1, \sqrt{2})$. The set E_r and its pre image are shown in red.

In this figure, given $r > 0$, the points where the capture circle of radius r intersects the vertical slit $[-i, i]$ have preimages $\tau_1, \tau_2, \bar{\tau}_1, \bar{\tau}_2 \in C_0$, where $\tau_1 = e^{i\phi}$ and $\tau_2 = e^{i\psi}$ with $\psi > \phi$. We also have that the preimage of the basepoint z_0 is $\zeta_0 = 1 - \sqrt{2} \in D_\zeta$. Now to compute $h(r)$, we consider the Cayley-type map

$R_0(\zeta,\tau_1,\tau_2,\bar{\tau}_1,\bar{\tau}_2) = \frac{\tau_2}{\tau_1}\frac{(\zeta-\tau_1)(\zeta-\bar{\tau}_2)}{(\zeta-\bar{\tau}_1)(\zeta-\tau_2)}$, where $\tau_1 = e^{i\phi}$ and $\tau_2 = e^{i\psi}$. Here, the angle ϕ varies from 0 to $\pi/2$, while the angle ψ varies from π to $\pi/2$. This map $R_0(\zeta)$ transforms the interior of the unit disc onto the lower halfplane and the boundary of the disc maps to the real line. Since $R_0 = \bar{R}_0$ for $|\zeta| = 1$, we have that $arg\, arg\, R_0 = k_0\pi$, where $k_0 \in Z$. By defining a potential function, $W_0(\zeta,\tau_1,\tau_2,\bar{\tau}_1,\bar{\tau}_2) = \frac{1}{\pi}[log\, log\, R_0\, + i\pi]$, we obtain $h(r) = Im[W_0(\zeta_0,\tau_1,\tau_2,\bar{\tau}_1,\bar{\tau}_2)]$ for $r \in [1,\sqrt{2})$.

This formula provides the same h-function graph as obtained in Figure 2.

<u>*Method 3*</u>: In this method, we explain the h-function computation by using the parallel-slit mapping which is in terms of the prime function.

Consider the interior of the unit disc as bounded circular domain D_ζ. Then the parallel-slit mapping $f(\zeta)$ transforms the domain D_ζ onto the complement of a vertical slit. Since the slit is symmetric about the real axis, the parameter α is real. Now, by multiply the expression (B) by the constant -½ to obtain the vertical sit with endpoints $i, -i$. Thus,

$$f(\zeta) = -\frac{1}{2}\left[\frac{1}{\alpha} + \frac{1}{\zeta-\alpha} + \frac{1}{\alpha(\alpha\zeta-1)}\right]$$

maps the domain D_ζ to the complement of the slit $[-i, i]$, where $\alpha = 1 \times 10^{-6}$. Since $f(\zeta_0) = z_0 = 1$, its preimage $\zeta_0 \approx -0.41421$ by using single-variable Newton's method. Moreover, $f(-i) = -i$ and $f(i) = i$. See Figure 4.

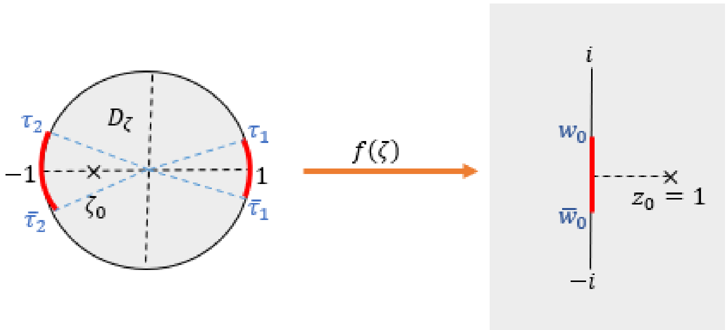


Figure 4: Conformal map from interior of unit disc to domain Ω with basepoint $z_0 = 1$, for regime $r \in [1,\sqrt{2})$. The set E_r and its pre image are shown in red.

The discussion for the computation of h-function is identical to that presented in Method 2. Only the conformal mapping formula is different here, since we use the parallel-slit mapping instead of the Joukowski mapping. The formulas for the Cayley-type map $R_0(\zeta,\tau_1,\tau_2,\bar{\tau}_1,\bar{\tau}_2)$ and the function $W_0(\zeta_0,\tau_1,\tau_2,\bar{\tau}_1,\bar{\tau}_2)$ hold in this case as well. Then, the h-function is given by $Im[W_0]$ evaluated at ζ_0. Thus, the h-function plot is identical to that shown in Figure 2.

CONCLUSIONS/RECOMMENDATIONS

In this study, the h-function in both explicit and implicit form for the unbounded simply connected region obtained by deleting a slit along the imaginary axis from the complex plane, has been cross-verified via three different methods. All these three methods produced identical h-function graphs. In future, we are interested to investigate the asymptotic behaviour of the h-functions that are given in implicit form confirming the consistency of the results.

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