TAILED QUANTUM SEARCH ON CYCLIC GRAPHS

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Quantum search algorithms on graphs are generally faster than their classical counterparts. Grover dynamics, a widely used quantum walk operator in search algorithms, works extremely well for many classes of graphs when searching for a marked vertex. However, due to the properties of the Grover dynamics, it does not perform well on cyclic graphs. In contrast, the tailed model of quantum walks guarantees the convergence of the walker's state, whereas in standard quantum walks, convergence is not guaranteed. Literature indicates that, similar to the usual quantum search algorithms, the tailed quantum search also performs well for complete graphs, with a finding probability of approximately 0.5 in the long run. In our work, we propose using Grover dynamics within the tailed model for searching on cyclic graphs. We demonstrate the quantum search on the smallest cyclic graphs, and we show through computation that the quantum search works effectively on cyclic graphs with up to 5 vertices using the tailed model. Furthermore, we show that the finding probabilities of a marked vertex in these cyclic graphs exceed 0.5.

Keywords: Quantum walk, Quantum search, Grover walk, Tailed model, Cyclic graphs

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INTRODUCTION

Quantum computing is one of the rapidly developing fields of study. Quantum walks is a tool in quantum computing used to address certain computational problems, such as the search problem, the element distinctness problem, and the network characterization problem [7, 8]. Quantum walks are a quantum analogue of classical random walks, where a quantum dynamic is applied to accelerate the algorithm compared to classical algorithms. Grover's algorithm is one such quantum search algorithm that searches a given item in an array of $\boldsymbol{N}$ data elements with high accuracy in $\boldsymbol{O}(\sqrt{\boldsymbol{N}})$ steps, whereas a classical algorithm would require $\boldsymbol{O}(\boldsymbol{N})$ steps [2]. A similar algorithm can be used to search for a particular vertex, called the marked vertex, on graphs where the dynamics of the algorithm is defined using Grover dynamics on each vertex. Compared to the classical search algorithms on graphs which have the run time $\boldsymbol{O}(\boldsymbol{N})$, the Grover search algorithm on graphs has a quadratic speed up with the run time $\boldsymbol{O}(\sqrt{\boldsymbol{N}})$ [5].

The Grover search algorithm works well on most classes of graphs, such as lattices, hypercubes [5] and Johnson graphs [6]. However, due to the properties of the Grover dynamics, the search does not perform well on cyclic graphs. To address this problem, we altered our model by connecting semi-infinite length paths, called tails, to all the vertices of the cyclic graph. The tailed model of quantum walks was first described in [1], which demonstrates quantum scattering. One important characteristic of the tailed model is the convergence of the state of the walk, while in the usual Grover model, the convergence of the state is not guaranteed [4]. Utilizing this property of convergence, it has been proved that the quantum search performs well for complete graphs in the convergent state [3, 9]. In our work, we describe a quantum search using the tailed model on cyclic graphs. As a trial, we compute the probability of finding a marked vertex on the cyclic graphs with up to 5 vertices.

METHODOLOGY

Let us denote the cyclic graph with $\boldsymbol{n}$ vertices by $\boldsymbol{C_n} = (\boldsymbol{V}, \boldsymbol{E})$, where $\boldsymbol{V}$ and $\boldsymbol{E}$ represent the sets of vertices and edges, respectively. Connect semi-infinite length paths to all the vertices. We adopt the setting of the walk from [3], where each edge in the graph is replaced by two symmetric arcs with opposite directions and denote the set of arcs by $\boldsymbol{A}$. For each arc $\boldsymbol{a} \in \boldsymbol{A}$ from the vertex $\boldsymbol{u}$ to the vertex $\boldsymbol{v}$, we denote $\boldsymbol{u}$, the origin of $\boldsymbol{a}$ by $\boldsymbol{o}(\boldsymbol{a}) = \boldsymbol{u}$ and $\boldsymbol{v}$, the terminus of $\boldsymbol{a}$ by $\boldsymbol{t}(\boldsymbol{a}) = \boldsymbol{v}$. We denote the inverse arc of $\boldsymbol{a}$ by $\overline{\boldsymbol{a}}$,where $\boldsymbol{o}(\boldsymbol{a}) = \boldsymbol{v}$ and $\boldsymbol{t}(\boldsymbol{a}) = \boldsymbol{u}$. Let the marked vertex be $\boldsymbol{w}$ and the state of the walker be a function $\boldsymbol{\phi} \in \mathbb{C}^{\boldsymbol{A}}$ with $\left|\left|\boldsymbol{\phi}\right|\right|^{\boldsymbol{2}} = \boldsymbol{1}$, where $\boldsymbol{\phi}(\boldsymbol{a})$ denotes the probability amplitude on the arc $\boldsymbol{a}$. The time evolution operator $\boldsymbol{U}$ of the walk is defined by

$$U_{a,b} = \begin{cases} -\left(\dfrac{2}{\deg\big(o(a)\big)} - \delta_{a,\bar{b}}\right) & if\ o(a) = t(a) = w, \\ \left(\dfrac{2}{\deg\big(o(a)\big)} - \delta_{a,\bar{b}}\right) & if\ o(a) = t(a) \neq w, \\ 0 & otherwise. \end{cases}$$

Here $\delta_{x,y}$ is the Kronecker delta function defined by

$$\delta_{x,y} = \begin{cases} 1 & if\ x = y, \\ 0 & x \neq y. \end{cases}$$

Let us denote the state of the walker at time step t by ϕ_t, then for a given initial state ϕ_0, the state of the walker at time step t is given recursively by $\phi_t = U\phi_{t-1}$ where U is the time evolution operator of the walk. We choose the initial state such that every arc on the tails towards the cycle has a probability amplitude of 1, and the remaining probability amplitudes are 0. By observing the symmetry, we denote the probability amplitudes as follows:

$$\phi_t(a) = \begin{cases} a_t^{(l)} \text{ if } \mathrm{dist}(o(a), w) = l \text{ and } \mathrm{dist}(t(a), w) = l-1, \\ \bar{a}_t^{(l)} \text{ if } \mathrm{dist}(o(a), w) = l-1 \text{ and } \mathrm{dist}(t(a), w) = l. \end{cases}$$

RESULTS AND DISCUSSION

It is already known that the quantum search performs well on the smallest cyclic graph $\boldsymbol{C_3}$ when the tails are connected, as $\boldsymbol{C_3}$ is a complete graph and the search performs well on the complete graphs with connected tails [4]. We demonstrate the search algorithm on the cyclic graph with 5 vertices. The following figure shows the probability amplitudes at the time step $\boldsymbol{t}$.

When $\boldsymbol{n}$ is odd

Cyclic graph of 5; $\boldsymbol{C_5}$

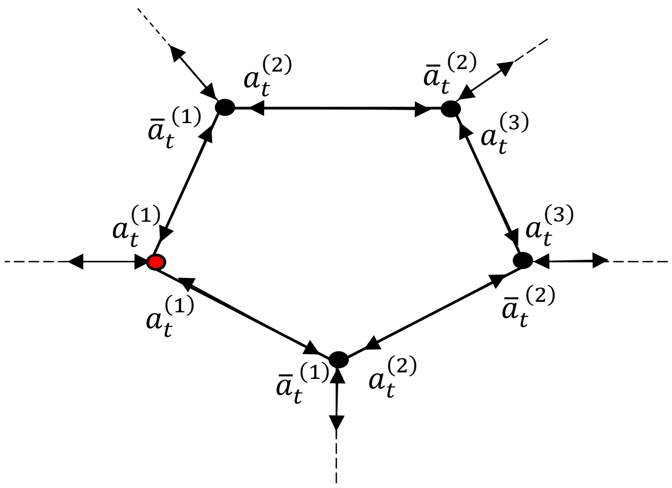


Figure 1

By the dynamics of the walk, we get the following system of recursions with the initial conditions $a_0^{(l)} = \bar{a}_0^{(l)} = 0$ for all l.

$$\boldsymbol{a_t^{(1)} = \frac{2}{3} - \frac{1}{3}\bar{a}_{t-1}^{(1)} + \frac{2}{3}a_{t-1}^{(2)}}$$
$$\boldsymbol{\bar{a}_t^{(1)} = -\frac{2}{3} - \frac{1}{3}a_{t-1}^{(1)}}$$
$$\boldsymbol{a_t^{(2)} = \frac{2}{3} - \frac{1}{3}\bar{a}_{t-1}^{(2)} + \frac{2}{3}a_{t-1}^{(3)}}$$
$$\boldsymbol{\bar{a}_t^{(2)} = \frac{2}{3} + \frac{2}{3}\bar{a}_{t-1}^{(1)} - \frac{1}{3}a_{t-1}^{(2)}}$$
$$\boldsymbol{a_t^{(3)} = \frac{2}{3} + \frac{2}{3}\bar{a}_{t-1}^{(2)} - \frac{1}{3}a_{t-1}^{(3)}}$$

By observing the coefficient matrix and the constant vector, we write the system

$$\varphi_t := \begin{pmatrix} a_t^{(1)} \\ \bar{a}_t^{(1)} \\ a_t^{(2)} \\ \bar{a}_t^{(2)} \\ a_t^{(3)} \end{pmatrix} = A\varphi_{t-1} + B$$

where $\boldsymbol{A} = \begin{bmatrix} 0 & -\frac{1}{3} & \frac{2}{3} & 0 & 0 \\ -\frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{3} & \frac{2}{3} \\ 0 & \frac{2}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$ and, $B = \begin{pmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$

Note that ϕ_t is the state of the walker at time step t while φ_t represents the vector of the probability amplitudes in the system above. We observe that the probability amplitudes in φ_t are contained within ϕ_t, and additionally, ϕ_t includes the repeating probability amplitudes. According to [4], $\lim_{t\to\infty} \phi_t$ exists and consequently $\lim_{t\to\infty} \varphi_t$ exists as well (denoted by φ_∞).
Taking limit

$$\lim_{t\to\infty} \varphi_t = \lim_{t\to\infty} A\varphi_{t-1} + B$$
$$\varphi_\infty = A\varphi_\infty + B$$
$$\varphi_\infty(I - A) = B$$
$$\varphi_\infty = (I - A)^{-1}B$$

By solving this system and scaling the in a similar way in [4], we get $\varphi_\infty = \frac{4}{\sqrt{189}} \begin{pmatrix} \frac{7}{3} \\ -\frac{5}{4} \\ 1 \\ -\frac{1}{2} \\ \frac{1}{4} \end{pmatrix}$

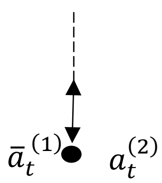
As shown in [4], the probability of finding w in the long run is given by

$$p(w) = \sum_{a:t(a)=w} |\phi_\infty(a)|^2 = 2\left|a_t^{(1)}\right|^2 = \frac{98}{189}.$$

We emphasize that the probability of finding the marked vertex is greater than 0.5, which confirms that the quantum search on C_5 works well with the tailed model.

When $\boldsymbol{n}$ is even

Cyclic graph of 4; $\boldsymbol{C_4}$



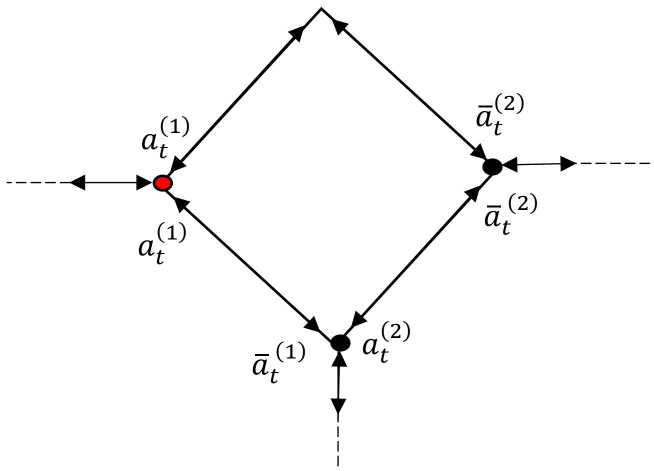


Figure 2

By the dynamics of the walk, we get the following system of recursions with the initial conditions $a_0^{(l)} = \bar{a}_0^{(l)} = 0$ for all l.

$$\boldsymbol{a_t^{(1)}} = \frac{2}{3} - \frac{1}{3}\boldsymbol{\bar{a}_{t-1}^{(1)}} + \frac{2}{3}\boldsymbol{a_{t-1}^{(2)}}$$
$$\boldsymbol{\bar{a}_t^{(1)}} = -\frac{2}{3} - \frac{1}{3}\boldsymbol{a_{t-1}^{(1)}}$$
$$\boldsymbol{a_t^{(2)}} = \frac{2}{3} + \frac{1}{3}\boldsymbol{\bar{a}_{t-1}^{(2)}}$$
$$\boldsymbol{\bar{a}_t^{(2)}} = \frac{2}{3} + \frac{2}{3}\boldsymbol{\bar{a}_{t-1}^{(1)}} - \frac{1}{3}\boldsymbol{a_{t-1}^{(2)}}$$

By observing the coefficient matrix and the constant vector, we have the system

$$\varphi_t := \begin{pmatrix} a_t^{(1)} \\ \bar{a}_t^{(1)} \\ a_t^{(2)} \\ \bar{a}_t^{(2)} \end{pmatrix} = A\varphi_{t-1} + B$$

where $A = \begin{bmatrix} 0 & -\frac{1}{3} & \frac{2}{3} & 0 \\ -\frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} \\ 0 & \frac{2}{3} & -\frac{1}{3} & 0 \end{bmatrix}$ and, $B = \begin{pmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$

By solving this system and scaling similar to the search in C_5, we get $\varphi_\infty = \frac{7}{4\sqrt{23}} \begin{pmatrix} \frac{10}{7} \\ -\frac{8}{7} \\ \frac{4}{7} \\ -\frac{2}{7} \end{pmatrix}$

The probability of finding w in the long run is given by

$$p(w) = \sum_{a:t(a)=w} |\phi_\infty(a)|^2 = 2\left|a_t^{(1)}\right|^2 = \frac{25}{46}.$$

We emphasize again that the probability of finding the marked vertex is greater than 0.5, which confirms that the quantum search on C_4 works well with the tailed model.

CONCLUSIONS/RECOMMENDATIONS

The standard Grover search algorithm does not perform well on cyclic graphs. To address this issue, we use the tailed model for the search algorithm. We have demonstrated that the quantum search algorithm works effectively on C_5. Intuitively, it can be expected that quantum search will work well on cyclic graphs in general. Furthermore, our model shows that, in the long run, the probability of finding the marked vertex exceeds 0.5. The model is more efficient when, for a given finding probability, the minimum run time of the algorithm is known, which indicates when the measurement of the state should be made to achieve the desired finding probability.

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