A NEW FOURTH-ORDER FINITE DIFFERENCE APPROXIMATION FOR FISHER KOLMOGOROV-PETROVSKY-PISKUNOV EQUATION

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Fisher Kolmogorov-Petrovsky-Piskunov (KPP) Equation (FE) is a nonlinear partial differential equation which is used to model physical scenarios involving the effects of both linear diffusions and nonlinear reactions. The FE arises in numerous applications, including brain tumor dynamics, population dynamics, chemical reactions, etc. Existing analytical methods for the FE often fall to give closed solutions and thus computational techniques have more commonly been used to attain a solution to the FE. The finite difference approximation (FDA) has widely been used to obtain a discrete solution to the FE. Combined with Crank-Nicholson (CN) technique, some second and fourth-order accurate CNFDAs appear in the literature too. This study aims to construct a fourth-order finite difference scheme for the FE. To achieve this, first, a pre-conditioned operator (P_h) for the second is derived. Acting P_h on the second derivative, a fourth-order approximation is obtained for the pre-conditioned second derivative. Second, the non-linear part of the FE is linearized using the lagging technique. Third, acting P_h on the FE and using the CN technique, a new CN finite difference scheme is derived for the FE. The preceding CN scheme is fourth-order accurate in space (with grid size h) and first-order accurate in time (with grid size τ). Furthermore, choosing $\tau = h^4$, a fully fourth-order accurate CN is obtained. Numerical results are obtained through numerical tests and compared with the corresponding results obtained from a recently derived fourth-order compact CN approximation. It demonstrates that the proposed fourth-order CN scheme is more accurate than the compact CN scheme.

Keywords: *Crank-Nicholson scheme, Lagging linearization, Preconditioned operator, Second order central difference approximation*

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INTRODUCTION

Fisher Kolmogorov-Petrovsky-Piskunov (KPP) equation is a nonlinear partial differential equation that is used to develop mathematical models of physical phenomena that characterize both linear diffusion and nonlinear reactions. The Fisher KPP equation arises in a variety of applications such as brain tumor dynamics, population dynamics, chemical reactions, etc. (Kenkre & Kuperman, 2003; Aronson & Weinberger, 1975; Belmonte-Beitia, et al. 2014). The one-dimensional Fisher KPP problem, in terms of time (t) and space (x) variables, is given by the initial boundary value problem:

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial t^2} + \alpha u(1-u), a < x < b, t \geq 0, \quad (1a)$$

$$u(x,0) = s(x), \qquad a \leq x \leq b, \quad (1b)$$

$$u(a,t) = b_1(t), u(b,t) = b_2(t), \qquad t > 0, \quad (1c)$$

where $\beta(> 0)$ is the diffusion coefficient, $\alpha(> 0)$is the reacting factor and $u(x,t)$ is the population density (Chemeda, et al. 2022).

Developing effective computational techniques for seeking solutions of the Fisher KPP problem has been an active area of research, due to the complexity of obtaining closed-form analytical solutions. The finite difference approximation (FDA) has been widely used in the literature to obtain discrete solutions for the Fisher KPP problem, compared to other methods, for example, (Al-Khaled, 2001; Mittal & Kumar , 2006; Rohila & Mittal, 2018). Among them, the Crank-Nicolson (CN) finite difference technique is a powerful technique to discretize the FE. Two implicit difference approximations for the FE using modified forms of CN and Keller Box methods have been proposed in the literature (Chandraker, Awasthi, & Simon, 2015). These implicit approximations show the second order accuracy in both time and space variables. Recently, a compact finite difference approximation has been derived for approximation (CCN) for the FE (Chemeda, etb al. 2022). This is first-order accurate in time and fourth order accurate in space. In the above FD's, the nonlinear term of FE is discretized to its corresponding linear form using the lagging technique.

In this study, we develop a new Crank-Nicholson finite difference scheme for the Fisher-KPP problem. This scheme is of the first order accurate in time and fourth order accurate in space. To attain this, we derive a preconditioned operator for the second order derivative and use it to compute the preconditioned second order derivative with an accuracy of order four from the second order accurate central difference approximation for the second derivative. The nonlinear term is linearized using the lagging technique.

METHODOLOGY

A new fourth order approximation

In this section, we derive a new fourth order approximation for the second derivative via a preconditioned operator. Using Taylor series with $D^k \equiv \frac{d^k}{dx^k}$, we obtain:

$$f(x+h) = f(x) + hDf(x) + \frac{h^2}{2!}D^2 f(x) + \frac{h^3}{3!}D^3 f(x) + \frac{h^4}{4!}D^4 f(x) + \cdots \quad (2)$$

$$f(x-h) = f(x) - hDf(x) + \frac{h^2}{2!}D^2 f(x) - \frac{h^3}{3!}D^3 f(x) + \frac{h^4}{4!}D^4 f(x) + \cdots \quad (3)$$

$(2) + (3) =>$

$$\frac{f(x+h) - 2f(x) + f(x-h)}{h^2} := D^2 f(x) + \frac{2h^2}{4!}D^4 f(x) + \frac{2h^4}{6!}D^6 f(x) + \frac{2h^6}{8!}D^8 f(x) + \frac{2h^8}{10!}D^{10} f(x) + \cdots$$

$$\Delta_h^2 f(x) = D^2 f(x) + \frac{2h^2}{4!}D^4 f(x) + \frac{2h^4}{6!}D^6 f(x) + \frac{2h^6}{8!}D^8 f(x) + \frac{2h^8}{10!}D^{10} f(x) + \cdots$$

$$= \left(I + \frac{h^2}{12}D^2\right)D^2 f(x) + c_1 h^4 + c_2 h^6 + O(h^8)$$

$$=: P_x D^2 f(x) + O(h^4), \text{as } h \to 0, \quad (4)$$

where $P_x = I + \frac{h^2}{12}D^2$ with identity operator I.

It is worthwhile to note that the accuracy order of the operator $P_x D^2$ depends upon that of the operator D^2 and thus Equation (4) does not guarantee precisely the accuracy order of $P_x D^2$ is 4. However, Equation (6) reveals that $P_x D^2 f(x)$ can be approximately computed from $\Delta_h^2 f(x)$. To obtain a fourth order accuracy for $P_x D^2$, we approximate the operator D^2 using the standard central difference approximation Δ_h^2 that gives a second order accuracy for D^2, as will be discussed in Theorem 1.

Theorem 1. *Let $f \in C^6(\Re)$. For a fixed $h > 0$, we have:*

$$\Delta_h^2 f(x) \equiv P_h D^2 f(x) + O(h^4), \quad (5)$$

where $P_h = 1 + \frac{h^2}{12}\Delta_h^2$.

Proof

$$P_x f(x) = \left(I + \frac{h^2}{12}D^2\right)f(x) = f(x) + \frac{h^2}{12}D^2 f(x) = \left(f(x) + \frac{h^2}{12}(\Delta_h^2 f(x) + O(h^2))\right)$$

$$= \left(I + \frac{h^2}{12}\Delta_h^2\right)f(x) + \left(\frac{h^2}{12} \times O(h^2)\right)$$

$$= \left(I + \frac{h^2}{12}\Delta_h^2\right)f(x) + O(h^4)$$

$$= P_h f(x) + O(h^4), \qquad (6)$$

where $P_h = I + \frac{h^2}{12}\Delta_h^2$. Equation (6) confirms that $P_x f(x)$ can be approximated with order 4 accuracy using the discrete operator P_h.

Now, replacing $f(x)$ by $D^2 f(x)$ in Equation (6) gives: $P_x D^2 f(x) = P_h D^2 f(x) + O(h^4)$. Then, from Equation (6), we have:

$$\Delta_h^2 f(x) = P_x D^2 f(x) + O(h^4) = \left(P_h D^2 f(x) + O(h^4)\right) + O(h^4)$$
$$= P_h D^2 f(x) + O(h^4) \qquad (7)$$

It is worth noting that using Equation (7), $P_h D^2 f(x)$ can be computed with order 4 accuracy from the central difference approximation $\Delta_h^2 f(x)$, which is the main concern of Theorem 1.

Corollary 1. *The operator P_h is linear.*

Application to Fisher Equation

To construct a CN scheme for the Fisher KPP equation, partition space and time domains into uniform partitions of size *N* and *M* with grid sizes $h = (b - a)/2$ and $\tau = T/M$, respectively. Also, let u_i^m be the numerical approximation to the exact solution $u(x_i, t_m)$, where $x_i = a + ih, t_m = m\tau$ for all $i = 0, 1, \cdots, N$ and $m = 0, 1, \cdots, M$. Acting P_h on Equation (1) and using Theorem (1) gives

$$P_h \frac{\partial u}{\partial t} = \beta P_h \frac{\partial^2 u}{\partial t^2} + \alpha P_h u(1-u) = \beta \Delta_h^2 u + O(h^4) + \alpha P_h u(1-u)$$

Using $\partial u(x_i, t_m)/\partial t = (u_i^{m+1} - u_i^m)/\gamma + O(\gamma^2)$ and the CN technique, one gets:

$$P_h \frac{u_i^{m+1} - u_i^m}{\gamma} = \frac{\beta}{2}\left(\Delta_h^2 u_i^{m+1} + \Delta_h^2 u_i^m\right) + P_h u_i^{m+1/2}\left(1 - u_i^{m+1/2}\right) + O(\tau) + O(h^4) \qquad (8)$$

Now using lagging technique, we linearize the nonlinear part of (1) as follows:

$$u_i^{m+1/2}\left(1 - u_i^{m+1/2}\right) = \frac{u_i^{m+1} + u_i^m}{2}\left(1 - \frac{u_i^{m+1} + u_i^m}{2}\right) = u_i^{m+1}(1 - u_i^m) + O(\tau) \qquad (9)$$

Now substituting (9) into (8), simplifying, and dropping the error term $O(\tau + h^4)$, we get the difference equation:

$$a_i^m u_{i+1}^m + b_i^m u_i^m + c_i^m u_{i-1}^{m+1} = a u_{i+1}^m + b u_i^m + c u_{i-1}^m, \qquad (10)$$

where
$a_i^m = 1 - 6\sigma - \alpha\tau(1 - u_{i+1}^m)$, $b_i^m = 10 + 12\sigma - 10\alpha\tau(1 - u_i^m)$, $c_i^m = 1 - 6\sigma - \alpha\tau\,(1 - u_{i-1}^m)$, $a = 1 + 6\sigma, b = 10 - 12\sigma$ and $c = 1 + 6\sigma$ with $\sigma = \beta\tau/h^2$.

RESULTS AND DISCUSSION

Example 1: $a = 0,\ b = 1,\ \beta = 1,\ s(x) = (1 + e^{x})^{-2},\ b_1(t) = (1 + e^{-5t})^{-2}, b_2(t) = (1 + e^{1-5t})^{-2}$ *and final time T=1* (Wazwaz, et al. 2004).

Results obtained for our proposed order 4 CN scheme (PCN) are compared with existing order 2 CN scheme (CN) and order 4 Compact CN scheme (CCN) in the literature (Chemeda, et al. 2022).

Table 1*: Comparison of maximum errors of C-N schemes for* $\tau = h$

h	Order 2 CN	Maximum errors	
		PCN	CCN
0.1	3.455e-03	3.466e-03	3.408e-03
0.05	1.829e-03	1.831e-03	1.821e-03
0.025	9.343e-04	9.345e-04	9.325e-04

Table 2*: Comparison of maximum errors of C-N schemes for* $\tau = h^4$.

h	Order 2 CN	Maximum errors	
		PCN	CCN
0.1	2.516e-06	3.776e-06	1.773e-05
0.05	7.872e-08	2.378e-07	5.198e-06
0.025	6.425e-08	1.485e-08	1.346e-06

Table 1 depicts the maximum absolute errors of order 2 CN, PCN, and CCN when $\tau = h$. The local truncation errors of 2 CN, PCN, and CCN are $O(h^2 + \tau) = O(\tau)$, $O(h^4 + \tau) = O(\tau)$, and $O(h^4 + \tau) = O(\tau)$,respectively. This may be the reason that all three methods show approximately the same accuracy. Table 2 demonstrates the maximum absolute errors of the three methods for $\tau = h^4$. In this case, the local truncation errors of 2 CN, PCN, and CCN are $O(h^2 + \tau) = O(h^2 + h^4) = O(h^2)$, $O(h^4 + \tau) = O(h^4)$, and $O(h^4 + \tau) = O(h^4)$, respectively. All the three schemes show better results than those results in Table 1. In fact, the proposed fourth-order scheme (PCN) is more accurate than the CCN. Except the truncation errors, there may be some factors that improve the accuracy of the second order CN scheme (Order 2 CN).

CONCLUSIONS

This study proposed a new fourth-order Crank-Nicholson difference scheme for the Fisher equation using a new pre-conditioned operator. Numerical results demonstrated the effect of our scheme. However, the lagging technique is first order accurate and thus a higher order accurate linearization is required to improve accuracy of the scheme considered in this study.

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